GYPSY FIELD PROJECT
IN RESERVOIR CHARACTERIZATION

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Objectives

The overall objective of this project is to use the extensive Gypsy Field laboratory and data set as a focus for developing and testing reservoir characterization methods that are targeted at improved recovery of conventional oil.

The Gypsy Field laboratory, as described by Doyle, O'Meara, and Witterholt (1992), consists of coupled outcrop and subsurface sites which have been characterized to a degree of detail not possible in a production operation. Data from these sites entail geological descriptions, core measurements, well logs, vertical seismic surveys, a 3D seismic survey, crosswell seismic surveys, and pressure transient well tests.

The overall project consists of four interdisciplinary sub–projects which are closely interlinked:

1. Modeling depositional environments.

2. Upscaling.

3. Sweep efficiency.

4. Tracer testing.

The first of these aims at improving our ability to model complex depositional environments which trap movable oil. The second entails testing the usefulness of current methods for upscaling from complex geological models to models which are more tractable for standard reservoir simulators. The third investigates the usefulness of numerical techniques for identifying unswept oil through rapid calculation of sweep efficiency in large reservoir models. The fourth explores what can be learned from tracer tests in complex depositional environments, particularly those which are fluvial dominated.

Summary of Technical Progress

During this quarter, the main activities involved the "Modeling depositional environments" sub–project, for which the progress is reported below.

1. Introduction. We consider a problem to estimate the permeability from core measurements and transient pressure data. Of particular interest is the dependence of the estimated permeability on pressure measurements. In this report we establish mathematical conditions under the estimated permeability is determined as a function of the pressure data that varies smoothly with respect to small changes in that data. This investigation is a key step in the study of the resolution properties of model-based estimation test problems.

2. The Semidiscrete Formulation of the Parabolic Problem. In this section we present the formulation for the parabolic models. To fix ideas, let Ω be an open bounded domain in $\mathbf{R}^n$ with a Lipschitz boundary $\partial\Omega$. Let $H = L^2(\Omega)$ and $V = H^1(\Omega)$. Let

$$f \in L^2(0, T; H) \text{ and } a \in Q \subset L^\infty(\Omega).$$

We assume that there is a positive constant ν such that

$$a(x) \geq \nu \text{ almost everywhere in } \Omega.$$

Consider the initial boundary value problem given by

$$(2.1) \qquad \frac{\partial u}{\partial t} - \nabla \cdot (a\nabla u) = f \text{ in } \Omega \mathrm{x}(0, T)$$

$$(2.2) \qquad \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega,$$

and

$$(2.3) \qquad u(\cdot, 0) = u_0 \in H$$

with $f \in L^2(0, T; H)$ and $a \in Q \subset L^\infty(\Omega)$. For ease we will take $u_0 = 0$. It is well known [2] that there exists a unique solution $u \in L^2(0, T; V)$. Furthermore, if $a_n \to a$ in Q for $a_n \geq \nu$, then the sequence of associated solutions $u(a_n)$ converges weakly to $u(a)$ in $L^2(0, T; H)$, [2]. In formulating a ROLS estimation problem, we suppose that Q is a Hilbert space that compactly imbeds into $L^\infty(\Omega)$.

We study systems of initial value problems obtained from the finite element approximations [2]. Suppose that $\{B_i\}_{i=1}^N$ and $\{b_i\}_{i=1}^M$ are linearly independent functions in U and Q, respectively. Express u and a as sums

$$u(t) = \sum_{i=1}^N c_i(t)B_i$$

and

$$a = \sum_{j=1}^M a_j b_j,$$

respectively. Given the coefficient a, we seek $u = u(a)$ such that

$$(2.4) \qquad \frac{\partial}{\partial t} \int_\Omega u(t) B_i dx \;+\; \int_\Omega a \nabla u(t) \cdot \nabla B_i dx \;=\; \int_\Omega f(t) B_i dx$$

for $i = 1, ..., N$. Introducing the representation of a as the above sum and collecting terms, we define component stiffness matrices as the N x N matrices $G^{(k)}$ with entries

$$G_{ij}^{(k)} = \int_\Omega b_k \nabla B_i \cdot \nabla B_j dx$$

for $k = 1, ..., M$ and

$$G_{0ij} = \int_\Omega B_i B_j dx.$$

Define the column N-vector valued function $t \mapsto F(t)$ with entries

$$F(t)_i = \int_\Omega f(t) B_i dx$$

for $i = 1, ..., N$, and set

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ \cdot \\ \cdot \\ \cdot \\ a_M \end{bmatrix}$$

and

$$c = c(t) = \begin{bmatrix} c_1(t) \\ c_2(t) \\ \cdot \\ \cdot \\ \cdot \\ c_N(t) \end{bmatrix}.$$

We also write $c = c(\mathbf{a})$ when it is desirable to emphasize the dependence of c on $\mathbf{a}$. The stiffness matrix is given as the linear combination of the component matrices

$$G = G(\mathbf{a}) = \sum_{k=1}^{M} a_k G^{(k)}.$$

and the discrete version of the boundary value problem (2.4) is thus given by the equation

$$(2.5) \qquad G_0 \frac{d}{dt} c \;+\; G(\mathbf{a}) c = F$$

with initial condition

$$(2.6) \qquad c(0) = 0$$

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Setting

$$S(a)(t) = \exp[G_0^{-1} G(a)t]$$

The solution to (2.5) may be represented by

(2.7) $$c(t) = \int_0^t S(\tau - t) G_0^{-1} F(\tau) d\tau.$$

Remark 2.1. The elliptic case is given by

$$G(\mathbf{a})c = F$$

where the vectors c and F no longer depend on t.

Suppose there are given continuous linear functionals $\{\Delta_n\}_{n=1}^{N_0}$ on V and $\{\Theta_n\}_{n=1}^{N_1}$ on Q to serve as observation functionals, [3]. From these functionals we construct the operators

$$C_0 : L^2(0, T; V) \mapsto Z_0 = L^2(0, T; \mathbf{R}^{N_0})$$

and $C_1 : Q \mapsto Z_1 = \mathbf{R}^{N_1}$ as

$$C_0 v(t) = \begin{bmatrix} < \Delta_1, v(t) > \\ \cdot \\ \cdot \\ \cdot \\ < \Delta_{N_0}, v(t) > \end{bmatrix}$$

and

$$C_1 \psi = \begin{bmatrix} < \Theta_1, \psi > \\ \cdot \\ \cdot \\ \cdot \\ < \Theta_{N_1}, \psi > \end{bmatrix},$$

respectively.

The minimization problem is formulated by introducing a fit-to-data functional

$$J(a) = \int_0^T \sum_{k=1}^{N_0} (< \Delta_k, u(t) > -z_k(t))^2 + \sum_{k=1}^{N_1} (< \Theta_k, a > -K_k)^2 +$$

$$+ \int_\Omega [\gamma_2 \mid D^2 a \mid^2 + \gamma_1 \mid \nabla a \mid^2] dx$$

where γ_1, and $\gamma_2 \geq 0$. The functional $J(a)$ is to be minimized over an admissible set $Q_{ad} \subset Q$. For example, Q_{ad} may be taken to be

(2.8) $$Q_{ad} = \{a \in H^2(\Omega) : a \geq \nu > 0\}.$$

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The finite dimensional formulation of the fit-to-data functional is obtained by introducing the N_0 x N matrix Φ

$$\Phi_{ij} = < \Delta_i, B_j >$$

for $i = 1, ..., N_0$ and $j = 1, ..., N$, the M x M matrix

$$H_{ij} = \int_\Omega [\gamma_1 \nabla b_i \cdot \nabla b_j + \gamma_2 D^2 b_i D^2 b_j] dx$$

for $i, j = 1, ..., M$, the N_1 x M matrix

$$\Psi_{ij} = < \Theta_i, b_j >$$

for $i = 1, ..., N_1$ and $j = 1, ..., M$, the N_0 column vector

$$\mathbf{z} = \begin{bmatrix} z_1 \\ \cdot \\ \cdot \\ \cdot \\ z_{N_0} \end{bmatrix},$$

the N_1 column vector

$$\mathbf{K} = \begin{bmatrix} K_1 \\ \cdot \\ \cdot \\ \cdot \\ K_{N_1} \end{bmatrix}.$$

Let

$$\Phi_2 = \Phi^* \Phi \text{ and } \Psi_2 = \Psi^* \Psi$$

where * denotes transposition. The functional $J(\cdot)$ may thus be viewed as being defined on $\mathbf{R}^M$ and is expressed as

(2.9) $$J(\mathbf{a}) = \int_0^T [c^* \Phi_2 c - 2z^* \Phi c + z^* z] dt + \mathbf{a}^* (H + \Psi_2) \mathbf{a} - 2\mathbf{K}^* \Psi \mathbf{a} + \mathbf{K}^* \mathbf{K}$$

where $\mathbf{a} \in Q_{ad}^M$ and Q_{ad}^M serves as an appropriate admissible set in $\mathbf{R}^M$.

To study the effect of perturbations of the data on interior optimal estimators, our starting point is the system of equations characterizing optimal estimators. Note the Frechet derivative of c at a with increment $\mathbf{a}'$, $Dc(\mathbf{a})\mathbf{a}'$, satisfies the equation

(2.10) $$G_0 \frac{d}{dt}[Dc(\mathbf{a})\mathbf{a}'] + G[Dc(\mathbf{a})\mathbf{a}'] = -G(\mathbf{a}')c(\mathbf{a}).$$

with initial condition

$$[Dc(\mathbf{a})\mathbf{a}'](0) = 0$$

so that

$$[Dc(\mathbf{a})\mathbf{a}'](t) = -\int_0^t S(\tau - t)G_0^{-1}G(\mathbf{a}')c(\mathbf{a})(\tau)d\tau.$$

Defining the column N-vectors

$$d_0^{(k)}(\mathbf{a})(t) = \int_0^t S(\tau - t)G_0^{-1}G^{(k)}c(\mathbf{a})(\tau)d\tau$$

and the N x M matrix

$$\mathcal{D}_0(\mathbf{a})(t) = [d_0^{(1)}(\mathbf{a})(t), ..., d_0^{(M)}(\mathbf{a})(t)],$$

we may write

$$[Dc(\mathbf{a})\mathbf{a}'](t) = -\mathcal{D}_0(\mathbf{a})(t)\mathbf{a}'.$$

It follows then that

$$(2.11) \quad \frac{1}{2}DJ(\mathbf{a})\mathbf{a}' = \int_0^T (\Phi_2 c(\mathbf{a}) - \Phi^* z)^* [Dc(\mathbf{a})\mathbf{a}'] \, dt + ((H + \Psi_2)\mathbf{a} - \Psi^* K)^* \mathbf{a}'$$

Introducing the vector $\pi = \pi(\mathbf{a}, \mathbf{z})$ as the solution of the system,

$$(2.12) \quad -\frac{d}{dt}G^{(0)}\pi + G(\mathbf{a})\pi = \Phi_2 c(\mathbf{a}) - \Phi^* z,$$

$$\pi(T) = 0,$$

we see that

$$\int_0^T (\Phi_2 c(\mathbf{a}) - \Phi^* z)^* G[Dc(\mathbf{a})\mathbf{a}'] \, dt = -\int_0^T \pi^* G(\mathbf{a}')c(\mathbf{a})dt$$

holds. The solution of (2.12) may be represented by the formula

$$\pi(\mathbf{a}, \mathbf{z})(t) = -\int_t^T S(t - \tau)G_0^{-1}(\Phi_2 c(\mathbf{a})(\tau) - \Phi^* z(\tau))d\tau.$$

Define the column M-vector $\mathcal{X} = \mathcal{X}(\mathbf{a}, \mathbf{z})$ with entries,

$$(2.13) \quad \mathcal{X}_k = \int_0^T \pi(\mathbf{a}, \mathbf{z})^* G^{(k)}c(\mathbf{a})dt.$$

The derivative of J may now be expressed by the formula

$$(2.14). \quad \frac{1}{2}DJ(\mathbf{a})\mathbf{a}' = [(H + \Psi_2)\mathbf{a} - \Psi^* K - \mathcal{X}]^* \mathbf{a}'$$

Thus, the optimality conditions statisfied by an interior solution are given by the following.

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Proposition 2.2. If a is an interior local minimum for the estimation problem, then **a** satisfies the system

$$(2.15)(i) \qquad G_0 \frac{d}{dt} c(\mathbf{a}) \ + \ Gc(\mathbf{a}) \ = \ F$$

$$c(\mathbf{a})(0) = 0$$

$$(2.15)(ii) \qquad -G_0 \frac{d}{dt} \pi(\mathbf{a}, \mathbf{z}) \ + \ G\pi(\mathbf{a}, \mathbf{z}) \ = \ \Phi_2 c(\mathbf{a}) - \Phi^* \mathbf{z}$$

$$\pi(\mathbf{a}, \mathbf{z})(T) = 0$$

$$(2.15)(iii) \qquad (H + \Psi_2)\mathbf{a} \ - \ \Psi^* \mathbf{K} \ - \ \mathcal{X}(\mathbf{a}, \mathbf{z}) \ = \ 0.$$

The optimality system in (2.15) establishes a relationship between the data vectors $\mathbf{z}$ and $\mathbf{K}$ and an optimal estimator $\mathbf{a}$. We next obtain conditions such that the relation given by the optimality conditions of Proposition 2.2 determines a function $\mathbf{z} \mapsto \mathbf{a}(\mathbf{z})$ from $\mathbf{R}^{N_0}$ into $\mathbf{R}^M$. To this end, define the function

$$\mathcal{F} : \mathbf{R}^M \ x \ Z_0 \ x \ \mathbf{R}^{N_1} \ \mapsto \mathbf{R}^M$$

by

$$(2.16) \qquad \mathcal{F}(\mathbf{a}, \mathbf{z}, \mathbf{K}) \ = \ (H + \Psi_2)\mathbf{a} \ - \ \Psi^* \mathbf{K} \ - \ \mathcal{X}(\mathbf{a}, \mathbf{z}).$$

For the time being we are interested only in the dependence of **a** on **z**. Hence, we view **K** as a constant vector and set

$$\mathcal{F}(\mathbf{a}, \mathbf{z}) \ = \ \mathcal{F}(\mathbf{a}, \mathbf{z}, \mathbf{K}).$$

Of course, existence of an interior solution for data **z** implies that the relation

$$(2.17) \qquad \mathcal{F}(\mathbf{a}, \mathbf{z}) \ = \ 0$$

holds. At a pair $(\mathbf{a}_0, \mathbf{z}_0)$ for which $\mathcal{F}(\mathbf{a}_0, \mathbf{z}_0) \ = \ 0$, the implicit function theorem asserts that if the Frechet partial derivatives, $D_a \mathcal{F}(\mathbf{a}_0, \mathbf{z}_0)$ and $D_z \mathcal{F}(\mathbf{a}_0, \mathbf{z}_0)$, of $\mathcal{F}$ exist and $D_a \mathcal{F}(\mathbf{a}_0, \mathbf{z}_0)^{-1}$ exists, then $\mathbf{z} \mapsto \mathbf{a}(\mathbf{z})$ is determined as a Frechet differentiable function in a neighborhood of $\mathbf{z}_0$, [1].

For any $\mathbf{a}'$, with $D_a \pi \ = \ D_a \pi(\mathbf{a}, \mathbf{z})$, and $Dc = Dc(\mathbf{a})$,

$$(2.18) \qquad -G_0 \frac{d}{dt}[(D_a \pi)\mathbf{a}'] + G(\mathbf{a})[(D_a \pi)\mathbf{a}'] \ = \ -G(\mathbf{a}')\pi \ + \ \Phi_2[(Dc)\mathbf{a}'].$$

and initial condition

$$[D_a \pi)\mathbf{a}'](T) \ = \ 0.$$

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Defining the N x M matrix $P(t)$ with columns

$$P_k(t) = \int_t^T S(t-\tau)G_0^{-1}G^{(k)}\pi(\mathbf{a},\mathbf{z})(\tau)d\tau$$

for k=1,...,M, and the N x M matrix

$$\mathcal{D}(t) = \int_t^T S(t-\tau)G_0^{-1}\Phi_2\mathcal{D}_0(\mathbf{a})(\tau)d\tau,$$

we may respresent $D_a\pi$ by the formula

$$[D_a\pi(\mathbf{a},\mathbf{z})\mathbf{a}'](t) = -(P(t)+\mathcal{D}(t))\mathbf{a}'$$

In addition, it easy to see that for $D_z\pi = D_z\pi(\mathbf{a},\mathbf{z})$

(2.19)
$$-G_0\frac{d}{dt}[(D_z\pi)\mathbf{z}'] + G(\mathbf{a})[(D_z\pi)\mathbf{z}'] = -\Phi^*\mathbf{z}'.$$

$$[(D_z\pi)\mathbf{z}'](T) = 0,$$

and

$$[(D_z\pi)\mathbf{z}'] = -\int_t^T S(t-\tau)G_0^{-1}\Phi^*\mathbf{z}'(\tau)d\tau.$$

It follows from equations (2.6) and (2.13) that

$$D_a\mathcal{X}_k(\mathbf{a},\mathbf{z})\mathbf{a}' = \int_0^T \{[D_a\pi(\mathbf{a},\mathbf{z})\mathbf{a}']^*G^{(k)}c(\mathbf{a}) + \pi(,\mathbf{z})^*G^{(k)}[Dc(\mathbf{a})\mathbf{a}']\}dt$$

and

$$D_z\mathcal{X}_k(\mathbf{a},\mathbf{z})\mathbf{z}' = \int_0^T [D_z\pi(\mathbf{a},\mathbf{z})\mathbf{z}']^*G^{(k)}c(\mathbf{a})dt.$$

Hence, we obtain the expressions

$$D_a\mathcal{X}_k(a,\mathbf{z})\mathbf{a}' = -[\int_0^T \{c(\mathbf{a})(t)^*G^{(k)}(P(t)+\mathcal{D}(t)) + \pi(\mathbf{a},\mathbf{z})(t)G^{(k)}\mathcal{D}_0(\mathbf{a})(t)\}dt]\mathbf{a}'$$

and

$$D_z\mathcal{X}_k(\mathbf{a},\mathbf{z})\mathbf{z}' = \int_0^T [\int_0^\tau c(\mathbf{a})(t)^*G^{(k)}S(t-\tau)dtG_0^{-1}\Phi^*]\mathbf{z}'(\tau)d\tau.$$

Setting
$$\mathbf{X}(\mathbf{a},\mathbf{z})(t) = -\begin{bmatrix} \int_0^t c(\mathbf{a})(t)^*G^{(1)}S(\tau-t)d\tau \\ \cdot \\ \cdot \\ \cdot \\ \int_0^t c(\mathbf{a})(t)^*G^{(M)}S(\tau-t)d\tau \end{bmatrix} G_0^{-1}\Phi^*,$$

we may write

$$D_z \mathcal{F}(\mathbf{a}, \mathbf{z})\mathbf{z}' = \int_0^T \mathbf{X}(\mathbf{a}, \mathbf{z})(t)\mathbf{z}'(t)dt$$

Furthermore, define the MxN matrices $\mathcal{K}_1$ and $\mathcal{K}_2$ in which the k-th rows are given by

$$\mathcal{K}_1 = \int_0^T c(\mathbf{a})(t)^* G^{(k)}(P(t) + \mathcal{D}(t))dt$$

and

$$\mathcal{K}_2 = \int_0^T \pi(\mathbf{a}, \mathbf{z})(t) G^{(k)} \mathcal{D}_0(\mathbf{a})(t)dt,$$

respectively, and set

$$\mathcal{K} = \mathcal{K}_1 + \mathcal{K}_2.$$

From (2.16), we see that

$$D_a \mathcal{F}(\mathbf{a}, \mathbf{z}) = H + \Psi_2 + \mathcal{K},$$

and from the implicit function theorem [1] we have the following.

Proposition 2.3. Suppose that $\mathcal{F}(\mathbf{a}_0, \mathbf{z}_0) = 0$. If matrix $H + \Psi_2 + \mathcal{K}$ is invertible, then there is a neighborhood $\mathcal{N}(\mathbf{z}_0)$ such that $\mathbf{z} \mapsto \mathbf{a}(\mathbf{z})$ is defined as a function on $\mathcal{N}(\mathbf{z}_0)$, and

$$(2.20) \qquad Da(\mathbf{z})\mathbf{z}' = (H + \Psi_2 + \mathcal{K})^{-1} \int_0^T \mathbf{X}(\mathbf{a}, \mathbf{z})(t)\mathbf{z}'(t)dt.$$

It is also of interest to calculate the second derivatives of $\mathbf{z} \mapsto \mathbf{a}(\mathbf{z})$. The following is a consequence of a straight forward calculation.

Lemma 2.4. The second derivatives of c and π satisfy the following equations.

$$(2.21)(i) \qquad G_0 \frac{d}{dt}[(D^2 c)(\mathbf{a}', \mathbf{a}')] + G(\mathbf{a})[(D^2 c)(\mathbf{a}', \mathbf{a}')] = -2G(\mathbf{a}')[(Dc)\mathbf{a}', (Dc)\mathbf{a}']$$

$$[(D^2 c)(\mathbf{a}', \mathbf{a}')](0) = 0$$

$$-G_0 \frac{d}{dt}[(D_{aa}^2 \pi)(\mathbf{a}', \mathbf{a}')] + G(\mathbf{a})[(D_{aa}^2 \pi)(\mathbf{a}', \mathbf{a}')] =$$

$$(2.21)(ii) \qquad = \Phi_2[(D^2 c)(\mathbf{a}', \mathbf{a}') - 2G(\mathbf{a}')[(D_a \pi)\mathbf{a}'],$$

$$[(D_{aa}^2 \pi)(\mathbf{a}', \mathbf{a}')](T) = 0$$

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$$-G_0 \frac{d}{dt}[(D_{az}\pi)(\mathbf{a}', \mathbf{z}')] + G(\mathbf{a})[(D_{az}\pi)(\mathbf{a}', \mathbf{z}')] = -\mathbf{G}(\mathbf{a}')[(D_z\pi)\mathbf{z}'], (2.21)(\text{iii})$$

$$[(D_{az}^2\pi)(\mathbf{a}', \mathbf{z}')](T) = 0$$

and

2.21(*iv*)
$$D_{zz}\pi = 0.$$

We note from (2.13) that the second Frechet derivative of $\mathcal{X}_k$ is given by

$$D_{zz}\mathcal{X}_k(\mathbf{a}, \mathbf{z})(\mathbf{z}', \mathbf{z}') = \int_0^T [D_{zz}\pi(\mathbf{a}, \mathbf{z})(\mathbf{z}', \mathbf{z}')]^* G^{(k)} c(\mathbf{a}) dt$$

Hence, by (2.21)(iv)

$$D_{zz}\mathcal{X}_k(\mathbf{a}, \mathbf{z}) = 0$$

from (2.16), we see that

$$D_{zz}\mathcal{F}(\mathbf{a}, \mathbf{z}) = 0.$$

Other second partial derivatives of $\mathcal{F}$ may be calculated similarly. From (2.16) and (2.17), we see that

Proposition 2.5. The second derivative of $\mathbf{z} \mapsto \mathbf{a}(\mathbf{z})$ with respect to $\mathbf{z}$ is given by

$$D^2 a(\mathbf{z})(\mathbf{z}', \mathbf{z}') = -(H + \Psi_2 + \mathcal{K})^{-1}\{D_{aa}\mathcal{F}(\mathbf{a}, \mathbf{z})(\mathbf{D}a(\mathbf{z})\mathbf{z}', \mathbf{D}a(\mathbf{z})\mathbf{z}') +$$

$$+ 2D_{za}\mathcal{F}(\mathbf{a}, \mathbf{z})(\mathbf{z}', Da(\mathbf{z})\mathbf{z}')\}.$$

Remark 2.6. Extending the above argument, it easy to see that if $H + \Psi_2 + \mathcal{K}$ is invertible, then any derivative of $\mathbf{a}$ exists.

Suppose that $\mathbf{z}'$ is such that $Da(\mathbf{z})\mathbf{z}' = 0$. That is, denoting the null space of $Da(\mathbf{z})$ by $N(Da(\mathbf{z}))$, suppose that

$$\mathbf{z}' \in N(Da(\mathbf{z})).$$

From Propositions 2.3 and 2.6, we have the following.

Corollary 2.7. If $\mathbf{z}'$ is such that $\mathbf{z}' \in N(Da(\mathbf{z}))$, then

$$D^{(n)}\mathbf{a}(\mathbf{z})(\mathbf{z}', ..., \mathbf{z}') = 0$$

for any n. For such vectors $\mathbf{z}'$ we see that

$$\mathbf{a}(\mathbf{z} + \mathbf{z}') = \mathbf{a}(\mathbf{z}).$$

Remark 2.8. It follows that the estimated coefficient a(z) is insensitive to any perturbation (no matter how large) $z' \in N(Da(z))$. Note this is a consequence of the fact that the fit-to-data functional is quadratic in the data z.

We next examine sufficient conditions underwhich the matrix $H + \Psi_2 + \mathcal{K}$ is invertible. To this end we introduce the following assumptions. Recalling that $G_0, G(a), G^{(k)}$ for $k = 1, ..., M$ are NxN symmetric matrices and that H and Ψ_2 are MxM symmetric matrices, we suppose there positive real numbers β, μ_0, ν_0, μ_1, and μ such that

$(2.22)(i)$
$$H + \Psi_2 \geq \beta I$$

$(2.22)(ii)$
$$\mu_1 I \geq G_0 \geq \mu_0 I$$

$(2.22)(iii)$
$$G(a) \geq \nu_0 I$$

and for any $k = 1, ..., M$

$(2.22)(iv)$
$$G^{(k)} \geq \mu I$$

where I represents the identity matrix on $\mathbf{R}^N$ or $\mathbf{R}^M$ which ever is appropriate. From straight forward estimates, we obtain

$$\|c(\mathbf{a})\|_{L^2(0,T,\mathbf{R}^N)} \leq \frac{1}{\nu_0}\|F\|_{L^2(0,T,\mathbf{R}^N)}$$

$$\|\pi(\mathbf{a},\mathbf{z})(t)\|_{L^2(0,T,\mathbf{R}^N)} \leq \frac{1}{\nu_0}\|\Phi_2 c(\mathbf{a})(t) - \Phi^* \mathbf{z}\|_{L^2(0,T,\mathbf{R}^N)}$$

$$\|\pi(\mathbf{a},\mathbf{z})\|_{L^2(0,T,\mathbf{R}^N)} \leq \frac{|\Phi|}{\nu_0}\{\frac{|\Phi|}{\nu_0}(\|F\|_{L^2(0,T,\mathbf{R}^N)} + \|\mathbf{z}\|_{L^2(0,T,\mathbf{R}^{N_0})}$$

$$\|Dc(\mathbf{a})\mathbf{a}'\| \leq \frac{\mu}{\nu_0^2}\|F\||\mathbf{a}'|$$

$$\|D_a \pi(\mathbf{a},\mathbf{z})\mathbf{a}'\| \leq [\frac{\mu}{\nu_0^2}(\|\Phi_2 c(\mathbf{a}) - \Phi^* \mathbf{z}\|_{L^2(0,T,\mathbf{R}^N)} + \frac{\mu_0|\Phi|^2}{\nu_0^3}\|F\|_{L^2(0,T,\mathbf{R}^N)}|]|\mathbf{a}'|$$

It follows then that for

$$K(\mathbf{a},\mathbf{z},F) = \mu\{[\frac{\mu}{\nu_0^2}(\|\Phi_2 c(\mathbf{a}) - \Phi^* \mathbf{z}\|_{L^2(0,T,\mathbf{R}^N)} +$$

$$+ \frac{\mu_0|\Phi|^2}{\nu_0^3}\|F\|_{L^2(0,T,\mathbf{R}^N)}|][\frac{1}{\nu_0}\|F\|_{L^2(0,T,\mathbf{R}^N)}] +$$

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$$+ \, [\frac{1}{\nu_0}\|\Phi_2 c(\mathbf{a})(t) - \Phi^* \mathbf{z}\|_{L^2(0,T,\mathbf{R}^N)}][\frac{\mu}{\nu_0^2}\|F\|]\},$$

we have

$$\|D_a \mathcal{X}_k(\mathbf{a}, \mathbf{z})\mathbf{a}'\|_{L^2(0,T)} \; \leq \; K|\mathbf{a}'|$$

and

$$\|D_a \mathcal{X}(\mathbf{a}, \mathbf{z})\mathbf{a}'\|_{L^2(0,T,\mathbf{R}^M)} \; \leq \; K|\mathbf{a}'|$$

Proposition 2.9. If

$$K(\mathbf{a}, \mathbf{z}, F) \; < \; \beta,$$

then $H + \Psi_2 + \mathcal{K}$ is invertible.

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